ECS 332: Principles of Communications **2012/1**

HW Solution 7 — Due: N/A

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Problem 1. ***Optimal code lengths that require one bit above entropy***: The source coding theorem says that the Huffman code for a random variable X has an expected length strictly less than $H(X)+1$. Give an example of a random variable for which the expected length of the Huffman code is very close to $H(X)+1$.

Problem 2. Consider the AWGN channel:

$$Y = X + N$$

where $N \sim \mathcal{N}(0, \sigma_N^2)$. Assume that $X \perp\!\!\!\perp N$ and that X takes two values, a and $-a$. For the whole question, assume $a > 0$.

(a) In this part, use $a = 5$ and $\sigma_N = 3$.

The channel output Y is fed into a decision device (a comparator) which will compare the value of Y to 0. The output $\hat{X}$ of the decision (thresholding) device is determined by

$$\hat{X} = \begin{cases} a, & \text{if } Y \geq 0, \\ -a, & \text{if } Y < 0. \end{cases}$$

The whole system in shown in Figure 7.1.

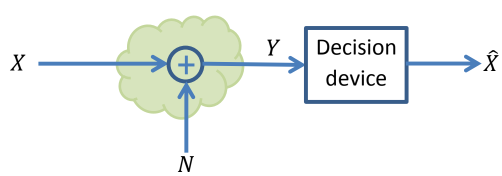

Figure 7.1: System for Q1.a

Find

(i) $P\left[\hat{X} = -a | X = a\right]$

(ii) $P\left[\hat{X} = a | X = a\right]$

(iii) $P\left[\hat{X} = -a|X = -a\right]$

(iv) $P\left[\hat{X} = a|X = -a\right]$

(v) $P\left[\hat{X} \neq X\right]$
Hint: By the total probability theorem,

$$\begin{aligned} P\left[\hat{X} \neq X\right] &= P\left[\hat{X} \neq X\,|X = a\right] P\left[X = a\right] + P\left[\hat{X} \neq X\,|X = -a\right] P\left[X = -a\right], \\ &= P\left[\hat{X} \neq a\,|X = a\right] P\left[X = a\right] + P\left[\hat{X} \neq -a\,|X = -a\right] P\left[X = -a\right]. \end{aligned}$$

(b) Continue from part (a).

We can use the system from part (a) to transmit/receive binary information by adding a simple mapping device with map 0 to a and 1 to $-a$ at the transmitter.

At the receiver, we also have another mapping device that map the a and $-a$ back to 0 and 1, respectively.

The new system is shown in Figure 7.2.

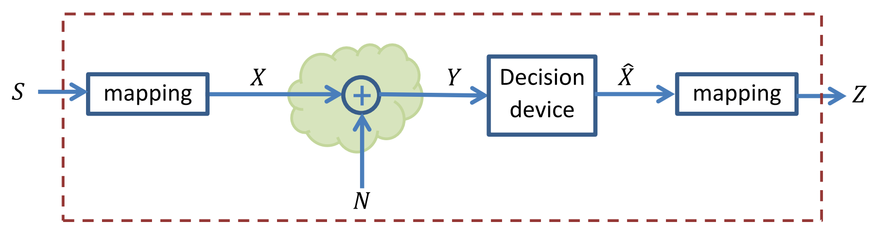

Figure 7.2: System for Q1.b

Note that S and Z are binary and that the whole system (inside the dotted box) in Figure 7.2 can be reduced to a binary symmetric channel (BSC) with crossover probability p.

Find p.

(c) Observe that as σ_N increases, the value of p in part (b) will also increase.

Is it possible to find σ_N such that $p > 0.5$?

(d) Express p using a, σ_N, and the Q function.

Problem 3. Consider a transmission over the BSC with crossover probability p. The random input to the BSC is denoted by S. Assume $S \sim \text{bernoulli}(p_1)$. Let Z be the output of the BSC.

(a) Suppose, at the receiver (which observes the output of the BSC), we learned that $Z = 1$. For each of the following scenarios, which event is more likely, $S = 1$ was transmitted or $S = 0$ was transmitted? (Hint: Use Bayes' theorem.)

(i) Assume $p = 0.3$ and $p_1 = 0.1$

(ii) Assume $p = 0.3$ and $p_1 = 0.5$

(iii) Assume $p = 0.3$ and $p_1 = 0.9$

(iv) Assume $p = 0.7$ and $p_1 = 0.5$

(b) Suppose, at the receiver (which observes the output of the BSC), we learned that $Z = 0$. For each of the following scenarios, which event is more likely, $S = 1$ was transmitted or $S = 0$ was transmitted?

(i) Assume $p = 0.3$ and $p_1 = 0.1$

(ii) Assume $p = 0.3$ and $p_1 = 0.5$

(iii) Assume $p = 0.3$ and $p_1 = 0.9$

(iv) Assume $p = 0.7$ and $p_1 = 0.5$

Remark: A MAP (maximum a posteriori) detector is a detector that takes the observed value Z and then calculate the most likely transmitted value. More specifically,

$$\hat{S}_{MAP}(z) = \arg\max_{s} P\left[S = s | Z = z\right]$$

In fact, in part (a), each of your answers is $\hat{S}_{MAP}(1)$ and in part (b), each of your answers is $\hat{S}_{MAP}(0)$.

Problem 4. Consider a repetition code with a code rate of 1/5. Assume that the code is used over a BSC with crossover probability $p = 0.4$.

(a) Assume that the receiver uses majority vote to decode the transmitted bit. Find the probability of error.

(b) Assume that the source produces source bit S with

$$P\left[S = 0\right] = 1 - P\left[S = 1\right] = 0.45.$$

Suppose the receiver observes 01001.

(i) What is the probability that 0 was transmitted? (Do not forget that this is a conditional probability. The answer is not 0.4 because we have some extra information from the observed bits at the receiver.)

(ii) What is the probability that 1 was transmitted?

(iii) Given the observed 01001, which event is more likely, $S = 1$ was transmitted or $S = 0$ was transmitted? Does your answer agree with the majority voting rule for decoding?

(c) Assume that the source produces source bit S with

$$P[S = 0] = 1 - P[S = 1] = p_0.$$

Suppose the receiver observes 01001.

(i) What is the probability that 0 was transmitted?

(ii) What is the probability that 1 was transmitted?

(iii) Given the observed 01001, which event is more likely, $S = 1$ was transmitted or $S = 0$ was transmitted? Your answer may depend on the value of p_0. Does your answer agree with the majority voting rule for decoding?

Problem 5. A channel encoder map blocks of two bits to five-bit (channel) codewords. The four possible codewords are 00000, 01000, 10001, and 11111. A codeword is transmitted over the BSC with crossover probability $p = 0.1$. Suppose the receiver observes 01001 at the output of the BSC.

(a) Assume that all four codewords are equally likely to be transmitted. Given the observed 01001 at the receiver, what is the most likely codeword that was transmitted?

(b) The Hamming distance between two binary vectors is defined as the number of positions at which the corresponding bits are different. For example, the Hamming distance between 00000 and 01001 is 2.

The minimum distance decoder is defined as the decoder that compare the Hamming distances between the observed bits at the receiver and each of the possible codewords. The output of this decoder is the codeword that give the minimum distance.

Explain why the minimum distance decoder would give the same decoded codeword as the decoder in part (a).

(c) What is the minimum (Hamming) distance d_{min} among codewords?

(d) Assume that the four codewords are not equally likely. Suppose 11111 is transmitted more frequently with probability 0.7. The other three codewords are transmitted with probability 0.1 each.

Given the observed 01001 at the receiver, what is the most likely codeword that was transmitted?